Zeno's First Argument

Motion is impossible, said Zeno, because a moving object must first go half the total distance it will travel, then half the remaining distance, and so on, forever. If a point moves from position 0 to position 1 on the number line, it first reaches position 1/2, then position 3/4, then position 7/8, and so on. At the nth stage, it is at position $1 - \frac{1}{2^n}$. From the fact that there is no n such that $1 - \frac{1}{2^n} = 1$, it follows that that moving point never reaches position 1. It just cannot get through the infinite number of stages necessary to do so. Hence there is no motion, motion from 0 to 1 being typical of any motion whatsoever.

In modern physics, we counter this argument by asserting that, indeed, the point can and does traverse each of the infinite number of intervals from $1 - \frac{1}{2^n}$ to $1 - \frac{1}{2^{n+1}}$ for $n = 1, 2, 3, \ldots$ — *ad infinitum*. There is no n such that the moving point does not cross position $1 - \frac{1}{2^n}$. Starting from the premiss that there is motion, modern physicists invoke the infinite to explain it. Like Zeno, they assume that motion is continuous, but, unlike Zeno, they are willing to say that a moving object does pass over an infinite number of points. Zeno rejected the infinite, and so he rejected motion too. Modern physicists accept motion, and so they accept the infinite too.

Zeno's Second Argument

The famous runner Achilles and his rival (usually thought to be a tortoise) are racing along the positive number line. Achilles starts at position 0, but the tortoise has a head start, beginning at position 1. Since Achilles runs twice as fast as the tortoise, one might expect him to overtake the tortoise at position 2. However, when Achilles arrives at position 1, the tortoise is already at position $1 + \frac{1}{2}$; when Achilles reaches position $1 + \frac{1}{2}$, the tortoise has raced on to position $1 + \frac{1}{2} + \frac{1}{4}$; and so on. When Achilles finally gets to position $2 - \frac{1}{2^n}$, for large n, the tortoise is still ahead, at position $2 - \frac{1}{2^{n+1}}$. Despite the appearances, which lead us to believe there is motion, Achilles will never catch up to the tortoise.

In this second argument, Zeno again assumed, as we do, that space and time are continuous, and that, if there is motion, there is uniform motion. Zeno also assumed, unlike us, that Achilles and the tortoise can never 'get through' the infinite number of stages into which Zeno analysed their motion.

For modern physics, precisely, motion typically consists of the occupation of infinitely many distinct locations at infinitely many distinct instants — all within a finite time interval. Because we accept the infinite, we do not find Zeno's argument troubling. However, if someone rejected the infinite, he or she would, indeed, have to reject the possibility of continuous motion.

Zeno's Third Argument

At every instant, a flying arrow is in exactly one fixed place. Hence it does not really move.

To this argument we would reply that the fact that the arrow covers 0 distance in an instant does not imply that it covers 0 distance in an interval consisting of an infinite number of instants. As every calculus student learns, there are cases in which

$$0 \times \infty = 1$$

Zeno did not like the infinite, so he did not make this reply.

Zeno's Fourth Argument

This argument is open to various interpretations. One is the following. There are three rows of people:

$$\begin{array}{ccccc} & A & A & A & A \\ & B & B & B & B \ \rightarrow \\ \leftarrow & C & C & C & C \end{array}$$

The *A*s are stationary, the *B*s are moving to the right at top speed, and the *C*s are moving to the left at top speed. Relative to each other, however, the *B*s and *C*s are going at twice top speed, which is impossible. So there cannot be any motion.

In answer to this argument we can either challenge Zeno's finitist assumption that there is a top speed, or we can invoke the Theory of Special Relativity, which explains how the *B*s and *C*s can both be going at the speed of light relative to the *A*s and yet *not* be going faster than the speed of light relative to each other.

The General Form of Zeno's Arguments

Each of Zeno's arguments has the following form:

$$\begin{array}{l} \text{Rejection of the infinite} \\ \underline{+ \text{ other considerations (including the continuity of space)}} \\ \text{No motion} \end{array}$$

This form is logically equivalent to the form:

$$\frac{\text{Motion} \\ + \text{ other considerations (including the continuity of space)}}{\text{Acceptance of the infinite}}$$

Most of us accept the existence of motion and would sooner give up finitism than embrace the static reality of Parmenides. The modern physicist, for one, is quite happy to base the analysis of motion on the mathematician's real number system, accepting the existence of infinite sets of numbers.

Democritus

Democritus of Abdera (in north-east Greece) lived about 420 B.C. He claimed that everything is made up of tiny indestructible atoms. The number of these atoms, he said, is infinite, and the empty space containing them is also infinite.

Democritus was a determinist. He asserted that 'from infinite time back are foreordained by necessity all things that were and are and are to come'. In harmony with this, he also held that everything happens without purpose or design.

Commenting on the circular sections of a cone cut by planes parallel to its base, Democritus asked:

> Are they equal or unequal? For, if they are unequal, they will make the cone irregular as having many indentations, like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder and to be made up of equal, not unequal, circles, which is very absurd.

Exercises 8

1. In the Cartesian plane, let

$$\begin{aligned} A_n &= (\frac{1}{2^{4n}},\ 0) \\ B_n &= (0,\ \frac{1}{2^{4n+1}}) \\ C_n &= (\frac{-1}{2^{4n+2}},\ 0) \\ D_n &= (0,\ \frac{-1}{2^{4n+3}}) \end{aligned}$$

Consider the path $A_0B_0C_0D_0A_1B_1C_1D_1A_2B_2C_2\ldots$, where each pair of adjacent points is joined by a straight line. Draw the beginning of this path. Show that this path has length $\sqrt{5}$. If you go along it, all the way, where will you end up? How many turns will you have made by the time you get there?

2. Suppose that, at time $t \geq 0$, Achilles is at point

$$(2 - \frac{1}{2^{t-1}},\ t)$$

— in polar coordinates — and the tortoise is at

$$(2 - \frac{1}{2^{t}},\ t)$$

Where are they when $t = 2\pi$? How far apart are they at time t? Will Achilles ever catch the tortoise?

Challenge for Experts

1. Let $f(0) = 0$ and, otherwise, $f(t) = t\sin(1/t)$. Then f is continuous, and, as t goes from 0 to 1, the graph of f is infinitely long.

Essay Questions

1. Does Zeno inadvertently prove that there is an infinite?

2. How would you answer Democritus's question? Is Democritus thinking of a cone as an infinite number of circles, one on top of the other?